Decoherence Caused by Topology in a Time-Machine Spacetime*

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Non-relativistic quantum theory of non-interacting particles in the spacetime containing a region with closed time-like curves (time-machine spacetime) is considered with the help of the path-integral technique. It is argued that, in certain conditions, a sort of superselection may exist for evolution of a particle in such a spacetime. All types of evolution are classified by the number n defined as the number of times the particle returns back to its past. It corresponds also to the topological class $\mathcal{P}_n$ of trajectories of a particle. The evolutions corresponding to different values of n are non-coherent. The amplitudes corresponding to such evolutions may not be superposed. Instead of one evolution operator, as in the conventional (coherent) description, evolution of the particle is described by a family U_n of partial evolution operators. This is done in analogy with the formalism of quantum theory of measurements, but with essential new features in the dischronal region (the region with closed time-like curves) of the time-machine spacetime. Partial evolution operators U_n are equal to integrals K_n over the classes of paths $\mathcal{P}_n$ if the evolution begins and ends in the chronal regions. If the evolution begins or/and ends in the dischronal region, the integral K_n over the class $\mathcal{P}_n$ should be multiplied by a certain projector to give the partial evolution operator U_n. Thus defined partial evolution operators possess the property of generalized unitarity $\sum_n U_n^\dagger U_n = \mathbf{1}$ and multiplicativity $U_m(t'', t')\, U_n(t', t) = U_{m+n}(t'', t)$. In the last equation however one of the numbers m or n (or both) must be equal to zero. Therefore, the part of evolution containing repeated returning backward in time cannot be factorized: all backward passages of the particle have to be considered as a single act, that cannot be presented as gradually, step by step, passing through 'causal loops'. The (generalized) multiplicativity and unitarity take place for arbitrary time intervals including i) propagating in initial and final chronal regions (containing no time-like closed curves) or from the initial chronal region to the final one, and ii) propagating within the time machine (in the dischronal region), from the time machine to the final chronal region or from the initial chronal region to the time machine.

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1 Introduction

It is known that General Relativity predicts the possible existence of spacetimes with non-trivial topology. The most intriguing among them are spacetimes with a region (or regions) containing closed time-like curves (CTC's). These *time-machine spacetimes* will be considered in the present paper. It is not clear whether the laws of physics permit the development of CTC's in the course of the evolution of a spacetime from some reasonable initial conditions (see [1]-[3] and references therein). We will not concern this problem here supposing that this is possible. Instead of this, taking spacetime geometry *ad hoc*, we shall investigate evolution of quantum particles in this geometry.

We shall discuss only non-relativistic processes. In this case the spacetime can be divided into *chronal regions* (regions without CTC's) and *dischronal regions* (containing CTC's) by slices $t = \text{const}$. For simplicity we shall discuss here the spacetime with only one dischronal region. This dischronal region (located between time moments t_1 and t_2) is preceded, at $t < t_1$, and followed, at $t > t_2$, by chronal regions. We shall call the first chronal region (at $t < t_1$) the initial region, the second chronal region (at $t > t_2$) the final region, and the dischronal region (at $t_1 < t < t_2$) the *time machine*.

In a series of rather recent papers (see for example [4]-[11]) the question was analyzed whether the standard laws of physics (classical and quantum) can accommodate, in a reasonable manner, a spacetime with CTC's. In this paper we shall focus our attention on the problem of propagating non-relativistic non-interacting quantum particles in the spacetime with CTC's (the time-machine spacetime).

The main conclusion made in the paper is that, in the case when closed time-like lines exist, a sort of superselection (decoherence) may arise in certain conditions, i.e. the superposition principle may be partially violated. In this case the conventional unitary description is not applicable in the time-machine spacetime. Instead, description by a series of partial evolution operators, obeying the generalized unitarity condition, must be applied. The description of this type has been earlier worked out in quantum theory of

continuous measurements.

It is well known now that the superposition principle of quantum mechanics is restricted when a measurement is performed. One may say that a sort of superselection exists in the situation of measurement, forbidding to superpose the states corresponding to different superselection sectors. In the case of quantum measurements the superselection sectors correspond to different alternative measurement outputs. The term 'decoherence' is often applied to describe this situation [12]-[15].

The superselection (decoherence) takes a specific form if a continuous (prolonged in time) measurement is performed (see [16]-[18] and references therein). In the procedure of path integration, those paths that are connected with different alternative outputs α of the continuous measurement, cannot be summed up. Instead of this, the amplitude A_α of each of these alternatives must be calculated by summation of the paths corresponding to the alternative α. After this, the probability of each alternative α can be found as a square modulus of the corresponding amplitude A_α. The sum of all these probabilities should be equal to unity.

This is valid not only in the situation when the measurement is performed on purpose, but also when a measurement-like interaction of the system of interest with its environment takes place. The latter means that information is recorded in the environment about the state of the system or its evolution. This information may be described with the help of alternatives. Each alternative α is a class of states or a class of paths of the system. Different classes should be considered as classical (decohering) alternatives. Superposition of amplitudes corresponding to different alternatives α is forbidden.

In practice the situation of the type of a measurement arises each time when "macroscopically distinct" states (or ways of evolution) of the system exist. What states must be considered to be macroscopically distinct, depends on what environment the system has.

This dependence on environment may be illustrated by the well-known two-slit ex-

periment. Propagation of the particle through the first or the second slit suggests two alternatives. Usually these are considered as quantum ('coherent', 'interfering') alternatives. One cannot know what of this alternative is actually realized. In this case superposition of paths describing propagating through different slits is possible, leading finally to the well-known interference effect.

However this consideration is valid only when the slits are in vacuum or in such a medium which cannot distinguish between two alternatives. If the interaction with the medium (environment) records information in this medium about what slit the particle propagates through, the situation is quite different and the preceding consideration is incorrect. It is possible of course to consider the environment and its interaction with the particle explicitly. However the influence of the environment onto the particle may be taken into account implicitly if one consider two alternatives (corresponding to the slits) to be classical ('non-coherent', 'incompatible'). Superposing the paths passing through different slits is in this case forbidden. This is the basic idea of quantum theory of measurements. More complicated situations of continuous quantum measurements may be found in [18].

Propagating in a spacetime containing CTC's, a particle may travel backward in time. This means that it can pass through some time interval (dischronal region) twice or more. Let the number of times the particle returns to its past be n. Then we have different ways of propagation characterized by the number n (in fact, nothing else than a winding number arising from non-trivial topology of the time-machine spacetime).

The situations characterized by different numbers n are quite different from the physical point of view. For the given n the particle passes through the dischronal region (within the time machine) $n + 1$ times. An observer in the dischronal region (within the time machine) will see $n+1$ particles existing simultaneously. Different values of n correspond to different numbers of particles in the dischronal region.

However in non-relativistic quantum mechanics the number of particles is conserved,

and a superselection corresponding to this number usually arises. If it really arises, the states corresponding to different numbers of particles may not be superposed.

The superselection (decoherence) connected with the number of particles is not absolute. It takes place for a massive particle possessing a charge of some type (for example electric charge) or spin (spinor charge). Even for such particles the decoherence exists only "in normal conditions". The latter means that the environment (consisting for example of photons) distinguishes between the states with different numbers of particles. Then any superposition of such states will decohere very quickly so that the superpositions may be considered to be forbidden, not existing.

This may be invalid in special conditions. Superpositions of the states with different numbers of particles exist for example in the superfluid (liquid helium at low temperature). In this case the environment does not distinguish between different numbers of particles, and no superselection connected with the particle number exists.

In all considerations of the present paper we shall suppose that the conditions in the dischronal region of the time machine are "normal" in this sense, i.e. they lead to superselection. Then the number of particles (of the considered type) existing simultaneously in this region presents a classical alternative. If this number is equal to $n+1$, this means that the particle n times returned from the future to the past. The situations corresponding to different n should be treated as non-coherent.

This presumption means that for the environment (medium) within the time machine alternative ways of propagation of the particle corresponding to different n are macroscopically distinct. One may say that a sort of measurement is performed in this case, and the result of the measurement is characterized by the number n. According to what has been said above, the paths characterized by different winding numbers n should be considered as non-coherent ('decohering'). Their summation is forbidden. Only the paths corresponding to the same n may be summed up forming corresponding propagators and evolution operators U_n.[1] Propagators and evolution operators with different n should not

[1]For evolution into or from the dischronal region summing over the nth class of paths is insufficient,

be summed up.

Hence, in the case when the superselection of different n takes place, the "general" evolution operator $U = \sum_n U_n$ (equal to the sum over all paths) makes no sense, and the evolution of the particle in a time-machine spacetime must be described in terms of the 'partial evolution operators' U_n. We shall develop this way of description including proof of generalized unitarity $\sum_n {U_n}^\dagger U_n = \mathbf{1}$ and generalized multiplicativity $U_m(t'', t')\, U_n(t', t) = U_{m+n}(t'', t)$. The evolution in a chronal region, between two chronal regions and within the dischronal region (within the time machine) will be considered. The resulting theory will be compared with the conventional (coherent) description of evolution as given in [11].

The paper is organized in the following way. In Sect. 2 general information about propagators and evolution operators in conventional theory is given, and an analogue of this formalism is presented (taken from quantum theory of measurement) for the case when the superposition principle is restricted by superselection. In Sect. 3 the superselection sectors (coherency sectors) are defined corresponding to the topological classes of particle trajectories in the time-machine spacetime. On the basis of this definition in Sects. 4, 5 the generalized unitarity and in Sect. 6 the generalized multiplicativity of evolution in the time-machine spacetime is proven. In Sect. 7 the results obtained are compared with the conventional unitary description of evolution in the time-machine spacetime. Special kind of states 'trapped' within the time machine are described in Sect. 8. Sect. 9 contains short concluding remarks.

2 Propagators, Paths and Measurements

In the conventional non-relativistic quantum mechanics the propagator $K(t'', x''|t', x')$ is a probability amplitude for the particle to transit from the point x' at time moment t' to

and multiplication by a certain projector is needed to obtain U_n, see Sect. 5.

the point x'' at time moment t''. The propagator may be considered as a kernel of the evolution operator $U = U(t'', t')$:

$$\psi_{t''} = U(t'', t')\, \psi_{t'}, \quad \psi_{t''}(x'') = \int K(t'', x''|t', x')\, \psi_{t'}(x')\, dx'. \tag{1}$$

An explicit expression for the propagator is presented by the path integral,

$$K(t'', x''|t', x') = \int d[x] \exp\left(\frac{i}{\hbar} S[x]\right), \tag{2}$$

where integration is performed over all paths $[x] = \{x(t)|t' \le t \le t''\}$ connecting the points (t', x') and (t'', x''), and $S[x]$ is the action functional calculated along the path $[x]$.

The evolution operator (propagator) satisfies the equation

$$U(t'', t')\, U(t', t) = U(t'', t), \quad \int K(t'', x''|t', x')\, K(t', x'|t, x)\, dx' = K(t'', x''|t, x) \tag{3}$$

(for $t \le t' \le t''$) which we shall call the property of *multiplicativity*. Besides, the evolution should conserve scalar products of the states, thus the evolution operator should be *unitary* that may be written in terms of propagators as follows:

$$\int K^*(t'', x''|t', x')\, K(t'', x''|t, x)\, dx'' = \delta(x', x).$$

The formalism of propagators and evolution operators must however be modified if a measurement or an observation of the system is performed. Such a modification is suggested by quantum theory of measurements and particularly by quantum theory of continuous measurements (see [16]-[18] and references therein).

The main feature of the resulting formalism is that a set of classical alternatives α should be considered. These alternatives arise as different measurement outputs. However, for emerging the situation of this type, it is not necessary that a measurement or an observation be arranged on purpose. The only condition necessary for this is that some information about quantum system be recorded in classical form in its environment. In the case of the time machine, the topological classes of trajectories described in Sect. 3 will play the role of classical alternatives.

The alternatives α are classical in the sense that they are incompatible. Correspondingly, quantum amplitudes corresponding to different alternatives cannot be summed up: they are non-coherent. Particularly, instead of a single propagator $K(t'', x''|t', x')$ or a single evolution operator $U(t'', t')$ a series of partial propagators $K_\alpha(t'', x''|t', x')$ or partial evolution operators $U_\alpha(t'', t')$ are necessary for describing evolution of the system. The partial propagators have not to be summed up.

How the partial evolution operators may be used to describe the evolution of the measured system? This should be made in different ways in the case of selective situation (for example selective measurement) and non-selective one. Selective situation means that it is known what alternative α is realized. The evolution is described in this case by one partial propagator or evolution operator $U_\alpha(t'', t')$. The evolution law, in terms of the wave function (state vector) or the density matrix, is following:

$$\psi_{t'}^\alpha = U_\alpha(t', t)\, \psi_t, \quad \rho_{t'}^\alpha = U_\alpha(t', t)\, \rho_t\, (U_\alpha(t', t))^\dagger\,. \tag{4}$$

The non-selective situation means that it is not known what concrete alternative is realized. In this case one should sum up over all possible alternatives. However, dealing with classical alternatives, one must sum up probabilities, not amplitudes. This means that summing must be performed in the second of the formulas (4) resulting in the following evolution law:

$$\rho_{t'} = \sum_\alpha \rho_{t'}^\alpha = \sum_\alpha U_\alpha(t', t)\, \rho_t\, (U_\alpha(t', t))^\dagger\,. \tag{5}$$

The final density matrix $\rho_{t'}$ has to be normalized ($\mathrm{Tr}\rho_{t'} = 1$) for any normalized initial matrix ρ_t. This takes place if and only if the following condition is fulfilled:

$$\sum_\alpha (U_\alpha(t', t))^\dagger\, U_\alpha(t', t) = \mathbf{1}. \tag{6}$$

It may be called the *generalized unitarity condition*. This condition means conservation of probability provided the probability of the alternative α to belong to the set $\mathcal{A}$ of

alternatives is defined as follows (see [18]):

$$\mathrm{Prob}(\alpha \in \mathcal{A}) = \mathrm{Tr} \sum_{\alpha \in \mathcal{A}} \rho_{t'}^{\alpha} = \mathrm{Tr} \sum_{\alpha \in \mathcal{A}} U_\alpha(t', t)\, \rho_t\, (U_\alpha(t', t))^\dagger\,. \tag{7}$$

If the alternatives (measurements) $\alpha_{t'}^{t''}$ corresponding to different time intervals $[t', t'']$ may be considered, their multiplication may be introduced,

$$\beta_{t'}^{t''}\, \alpha_t^{t'} = \gamma_t^{t''}.$$

Then the following *multiplicative law* should be valid [18] for the corresponding evolution operators (propagators):

$$U(\beta_{t'}^{t''})\, U(\alpha_t^{t'}) = U(\gamma_t^{t''}), \quad t \leq t' \leq t''. \tag{8}$$

All these concepts, elaborated previously for continuous quantum measurements, may now be applied to the alternatives corresponding to different topological numbers n in the time-machine spacetime.

In the paper [11] the formalism of propagators in the form of path integrals has been applied to describe the evolution of a non-relativistic particle in the time-machine spacetime. The propagator for such a particle was defined as an integral over *all paths*. It was shown that, in a simple model of the time-machine spacetime, unitarity and multiplicativity for the propagator are violated if at least one of the time moments is in the dischronal time interval.

We argued in Sect. 1 that a sort of superselection may arise in the time-machine spacetime for topologically different paths. If this is the case, then the "general" propagator determined by integrating over all paths makes no sense. Instead, one must use *partial propagators* just as in theory of continuous quantum measurements. Each of partial propagators should be a sum over topologically equivalent paths.[2] Let us come over to description of the corresponding classes of paths and partial propagators.

[2]This is valid for the evolution from the past of the time machine to its future. For evolution into or from the time machine the sum over topologically equivalent paths should be multiplied by a certain projector, see Sect. 5.

3 Topological Classes of Paths

Let us consider the concrete model of a time-machine spacetime and define the topological classes of trajectories of a non-relativistic particle in such a spacetime. We shall discuss hereafter the model of a time-machine spacetime that has been used in the paper [11]. It will be evident though that at least some of the results have more general validity.

The spacetime under consideration may be constructed (see Fig. 1) from a usual, 'chronal' one by adding two temporary wormholes. The original chronal spacetime will be called the *background spacetime*. The wormholes added to the background spacetime connect two space regions S_1 and S_2 belonging to the time slices t_1 and t_2 correspondingly, with $t_1 \leq t_2$. One of the wormholes W_1 leads from the past region S_1 to the future region S_2. This wormhole is entered if the region S_1 is approached from its 'past side' (but not through the wormhole W_2). Another wormhole W_2 leads from the future region S_2 to the past region S_1 if the region S_2 is approached from its 'past side' (but not through the wormhole W_1). About the properties of walls of the wormholes see the paper [11].

Consider all paths of a non-relativistic particle in this spacetime and divide them in the topological classes. The classes will then be connected with non-coherent sectors (classical alternatives) in description of the particle evolution in the spacetime.

The first class is formed by the paths going through the 'future-directed' wormhole W_1, entering S_1 and going out from S_2. Let us denote this class by $\mathcal{P}_{00}$. The scheme of the path belonging to this class is following:

$$\mathcal{P}_{00}: \text{ initial point} \to S_1 \to W_1 \to \text{final point.}$$

One more class $\mathcal{P}_0$ includes the paths lying completely in the background spacetime. These paths bypass both entrances S_1 and S_2 to the wormholes. This is a class of topologically simple paths. The scheme of the paths of this class is following:

$$\mathcal{P}_0: \text{ initial point} \to \text{final point.}$$

Figure 1: The time-machine spacetime is constructed from an usual (chronal) background spacetime by attaching two wormholes connecting two space regions in two different time slices. The wormhole W_1 leads from the past to the future connecting the 'past side' of the region S_1 with the 'future side' of the region S_2. The wormhole W_2 leads from the future to the past connecting the 'past side' of the region S_2 with the 'future side' of the region S_1.

The class $\mathcal{P}_1$ contains the paths that pass through the wormhole W_2 once. The path of this type enters the mouth S_2 of the wormhole at time t_2 and go out of the mouth S_1 at time t_1. The scheme of these paths is following:

$$\mathcal{P}_1 : \text{ initial point} \to S_2 \to W_2 \to \text{final point.}$$

The path of the class $\mathcal{P}_2$ proceeds through the wormhole W_2 twice, going into S_2 at time t_2, going out of S_1 at t_1, proceeding through the background spacetime again to time t_2, once more entering S_2, again going out of S_1, and ultimately approaching the final point through the background spacetime. This corresponds to the following scheme:

$$\mathcal{P}_2 : \text{ initial point} \to S_2 \to W_2 \to S_2 \to W_2 \to \text{final point.}$$

Generally, the paths of the class $\mathcal{P}_n$ with $n \geq 0$ pass through the wormhole W_2 precisely n times, according to the scheme

$$\mathcal{P}_n : \text{ initial point } (\to S_2 \to W_2)^n \to \text{final point.}$$

It was argued in Introduction that the types of evolution corresponding to different numbers of returns of the particle to its past are usually (in "normal conditions") macroscopically distinct and therefore cannot be coherently superposed. These types of evolution correspond to the paths belonging to different topological classes. If the decoherence takes place, the propagator defined by integrating over all paths (as in Eq. (2)) makes no sense. It seems reasonable to consider instead the propagators obtained by integrating over the topological classes:

$$K_n(t'', x''|t', x') = \int_{\mathcal{P}_n} d[x] \exp\left(\frac{i}{\hbar} S[x]\right). \tag{9}$$

The operators corresponding to these two-point functions will be denoted by K_n:

$$\psi_{t''}(x'') = (K_n(t'', t')\, \psi_{t'})\,(x'') = \int K_n(t'', x''|t', x')\, \psi_{t'}(x')\, dx'.$$

If both the initial and final time moments t', t'' are in the same chronal region (before or after the time machine), then only one topologically trivial class $n = 0$ exists, and only one operator K_0 is defined by Eq. (9). It is evident that, according to general theory (Sect. 2), this operator $K_0 = U_0 = U$ is the evolution operator of the particle in the chronal region. This operator is unitary.

If the time moments t', t'' are in different chronal regions or some of them (or both) are in the dischronal region, then there are operators K_n corresponding to arbitrary $n = 00, 0, 1, 2, \ldots$. They seem to be good candidates for describing evolution in the nth superselection sector (nth classical alternative). We shall see that this is valid in the case when the time t' is before the time machine emergence, $t' < t_1$ and the time t'' is after it disappearing, $t'' > t_2$. In this case the operators $K_n = U_n$ play the role of 'partial evolution operators' satisfying the generalized unitarity condition, $\sum_n {U_n}^\dagger U_n = 1$. However when the time moment t'' is in the dischronal region (within the time machine), then the partial evolution operators U_n will be shown to differ from the operators K_n by certain projectors (see Sect. 5).

This is only natural because of the following.

The coherency sectors for evolution of a particle in the time-machine spacetime may be defined as a number of times the particle returns to its past. The set of such superselection sectors is in one-to-one correspondence with the topological classes $\mathcal{P}_n$ of paths between the time $t' < t_1$ and the time $t'' > t_2$. Thus the above-defined operators K_n must coincide in the case $t' < t_1 < t_2 < t''$ with what we call 'partial evolution operators' U_n. However, if the final time t'' of the evolution is within the time machine, $t_1 < t'' < t_2$, the class $\mathcal{P}_n$ of paths leading to this time does not correspond to the nth coherency sector.

Indeed, after the time moment t'' the particle can either escape the time machine or enter the wormhole W_2 and return to its past one or more times. Therefore, the operator K_n (defined by summing up over the class $\mathcal{P}_n$) does not in fact coincide with the partial evolution operator for the nth alternative. If however we provide, multiplying K_n by the corresponding projector, that the particle escape the time machine after t'', then we have the correct partial evolution operator U_n.

4 Generalized Unitarity I

We considered in Sect. 2 the general situation when non-coherent sectors exist in the evolution of a quantum system. In Sect. 3 coherency sectors connected with topological classes of paths were defined in the time-machine spacetime. In the present section and the next one we shall formulate and prove the generalized unitarity condition for this case.

As has already been argued, the classical alternatives may be identified (at least for the case $t' < t_1 < t_2 < t''$) with the classes of paths $\mathcal{P}_n$ introduced in Sect. 3. Operators K_n were defined in Sect. 3 by summation over classes $\mathcal{P}_n$. It is natural to try and identify these operators with the partial evolution operators U_n corresponding to the alternatives. This turns out to be correct for the most important case $t' < t_1 < t_2 < t''$ (evolution from the past of the time machine to its future). This may be accepted also in the cases when both t', t'' are in the same chronal region. A more complicated situation arises if t'' is within the time machine (in the dischronal region). The corrections for this case will

be introduced in the next section.

Consider first the trivial situations of the evolution within one of two chronal regions. In the case $t' < t'' < t_1$ (as well as for $t_2 < t' < t''$) there is only one (trivial) class of paths with $n = 0$ and only one operator K_0. This operator coincides in fact with the evolution operator in the background spacetime. Evolution in this case does not differ from usual coherent (unitary) evolution and is described by a single evolution operator $U = U_0 = K_0$ (we omit hereafter the explicit specification of the initial and final time moments). The generalized unitarity condition coincides in this case with the conventional unitarity, $U^\dagger U = \mathbf{1}$.

If the evolution not restricted by a single chronal region is considered, all topological classes $n = 00, 0, 1, 2, \dots$ (and corresponding coherency sectors) exist. According to the general formula (6), the generalized unitarity condition must then have the form

$$U_{00}{}^\dagger U_{00} + \sum_{n=0}^{\infty} U_n{}^\dagger U_n = \mathbf{1} \tag{10}$$

where we omitted specification of the time interval. We shall see that this condition is actually fulfilled for the evolution operators K_n corresponding to the propagators (9) provided the final time of the evolution, $t'' > t_2$. This is why the operators K_n may actually be identified with the partial evolution operators U_n in this case.

The generalized unitarity condition (10), as usual for quantum mechanics, expresses conservation of probability. If an initial state is described by the density matrix ρ, then the probability of realization of nth alternative is, according to quantum theory of measurements (see Eq (7)),

$$P_n = \mathrm{Tr}\rho_n = \mathrm{Tr}\left(U_n \rho U_n{}^\dagger\right). \tag{11}$$

The condition (10) provides then that

$$P_{00} + \sum_{n=0}^{\infty} P_n = 1$$

for an arbitrary initial state ρ. The same probability interpretation of the generalized unitarity is valid for all other choices of t', t'' that will be considered below.

To demonstrate that the condition (10) is valid, we shall express the operators K_n in a more explicit form, through the evolution operators in the background chronal spacetime (we shall denote them by V_i) and the evolution operators in the wormholes (denote these operators W_i). Besides this, two projectors in the background spacetime (P_i) will be used describing entering the mouths of the wormholes. Expressions for the operators K_n may be readily constructed with the help of the description of classes $\mathcal{P}_n$ (see Sect. 3).

Consider first the case $t' < t_1 < t_2 < t''$, i.e. the evolution starting before the time machine emergence and finishing after it disappearance.

The paths of the class $\mathcal{P}_{00}$ go from the starting point to the point at time moment t_1, then enter the wormhole W_1, proceed through it to the moment t_2 and then go from t_2 to the final point. Summation over all paths of this type gives us the evolution operator K_{00} in the form of the product $V_2 W_1 P_1 V_1$ where V_1 and V_2 are the evolution operators outside the time machine (first of them between starting time t' and time t_1, and the second one between t_2 and the final time t''), and W_1 is the evolution operator along the wormhole W_1. The projector P_1 onto the region S_1 provides entering the particle into the wormhole.

Analogously the expressions for all operators K_n can be found in the case $t' < t_1 < t_2 < t''$. We shall see below that these operators satisfy the generalized unitarity condition and may be identified with the partial evolution operators. This is why we shall denote them by U_n:

$$U_{00} = K_{00} = V_2 W_1 P_1 V_1, \quad U_n = K_n = V_2(1-P_2)(V_{21} W_2 P_2)^n V_{21}(1-P_1)V_1, \quad n \geq 0. \tag{12}$$

Here V_{21} is the evolution operator in the background spacetime between time moments t_1 and t_2. The projector P_2 provides entering the region S_2 and therefore the wormhole W_2 while the complementary projector $1 - P_2$ provides bypassing the wormhole W_2. Analogously the projector $1 - P_1$ provides bypassing W_1.[3]

Now we can easily prove the generalized unitarity (10) for the operators $U_n = K_n$ defined by Eq. (12). In the proof we shall use unitarity of all V_i and W_i as well as the

[3]We denoted here the unity operator by 1 instead of **1**.

properties of projectors, $P_i^2 = P_i$ and $P_i{}^\dagger = P_i$. Let us present the expression $U_n{}^\dagger U_n$ in the form

$$\begin{aligned} U_n{}^\dagger U_n &= A^\dagger \left(B^\dagger\right)^n V_{21}{}^\dagger (1-P_2) V_{21} B^n A \\ &= A^\dagger \left(B^\dagger\right)^{n-1} C B^{n-1} A - A^\dagger \left(B^\dagger\right)^n C B^n A \end{aligned} \tag{13}$$

where

$$A = (1-P_1)V_1, \quad B = W_2 P_2 V_{21}, \quad C = V_{21}{}^\dagger P_2 V_{21}.$$

Summing (13) in n, we shall obtain

$$\sum_{n=1}^{\infty} U_n{}^\dagger U_n = A^\dagger C A = V_1{}^\dagger (1-P_1) V_{21}{}^\dagger P_2 V_{21} (1-P_1) V_1. \tag{14}$$

Adding two other terms $U_0{}^\dagger U_0$ and $U_{00}{}^\dagger U_{00}$, we shall prove finally the generalized unitarity condition (10).

Remark 1 In the proof given above (and in the analogous proofs hereafter) it is supposed implicitly that the terms of the sum (10) tend to zero when n tends to infinity. This is not valid in the subspace of some 'finely tuned' states, and the formula (10) is not applicable in this subspace. More precisely, this formula, as an equality of two operators, is valid in the space of states $|\psi\rangle$ for which

$$\lim_{n\to\infty} \langle\psi| A^\dagger \left(B^\dagger\right)^n C B^n A |\psi\rangle = 0$$

(in the notations introduced above). From the physical point of view, this means that the probability for the particle (in the considered state) to return n times into its past infinitely decreases when n tends to infinity. This is right for 'almost all' states. We shall see in Sect. 8 however that this is not valid for some finely tuned states trapped within the time machine.

Consider now the case when the initial time moment t' is within the interval $[t_1, t_2]$, but the final time t'' is after t_2. In this case the paths cannot belong to the class $\mathcal{P}_{00}$, thus

the generalized unitarity takes the form

$$\sum_{n=0}^{\infty} {U_n}^{\dagger} U_n = \mathbf{1}. \tag{15}$$

Let us prove it.

In the case $t_1 < t' < t_2 < t''$ we have for the operators K_n (that again will prove to coincide with the partial evolution operators U_n) the following formulas:

$$U_n = K_n = V_2(1 - P_2)(V_{21} W_2 P_2)^n V_{20}, \quad n \geq 0, \tag{16}$$

where V_{20} denotes the propagator in the background (chronal) spacetime from the initial time t' (between t_1 and t_2) to the time t_2. Summing up the expression for ${U_n}^{\dagger} U_n$ in the same way as above gives the generalized unitarity in the form (15) provided we accept the definition (16).

Let now the final time t'' of the evolution be between t_1 and t_2. We shall see that the (generalized) unitarity does not take place for the operators K_n in this case, so that these operators cannot be identified with partial evolution operators. Right form of the partial evolution operators will be given in Sect. 5.

Constructing the operators K_n in the same way as above (i.e. as integrals over the topological classes of paths), we are led to the following formulas for the case $t' < t_1 < t'' < t_2$ (when the initial time is earlier than the time machine emergence):

$$K_n = V_{01}(W_2 P_2 V_{21})^n (1 - P_1) V_1, \quad n \geq 0 \tag{17}$$

(V_{01} describes evolution between t_1 and the final time t''). If both initial and final times are within the time machine ($t_1 < t' < t'' < t_2$), we have

$$K_0 = V_{0'0}, \quad K_n = V_{0'1} W_2 P_2 (V_{21} W_2 P_2)^{n-1} V_{20}, \quad n \geq 1. \tag{18}$$

In both cases (17), (18) the generalized unitarity does not take place.

One can readily see that the algebraic operations used earlier to prove the generalized unitarity, are now impossible because there is no projector of the form $1 - P_2$ in the

expressions (17), (18). This gives a hint that the operator U_n must differ from K_n by the factor of the form $1 - Q_2$ where Q_2 is a projector. We shall see in Sect. 5 that this may be justified physically.

5 Generalized Unitarity II

It has been shown in the preceding section that the operators K_n defined with the help of the classes of paths $\mathcal{P}_n$ and expressed by the formulas (17), (18) do not satisfy the generalized unitarity condition if the evolution is considered to the time moment t'' between t_1 and t_2.The physical reason of this is that the classes of paths do not correspond in this case to the coherency sectors, i.e. to macroscopically distinguishable (distinct) physical situations. Therefore, the operators K_n are not in this case partial evolution operators.

Indeed, the situations are macroscopically distinct if they differ by the number of particles in the time interval $[t_1, t_2]$. If the path belongs to the class $\mathcal{P}_n$ and ends after t_2, it describes a quite definite number of particles, $n + 1$ (because returning to the past is impossible after time t_2). However, if the path ends in the point $t'' < t_2$, then the number of particles in the interval $[t_1, t_2]$ is indefinite, since it is not known whether the particle will escape the time machine after t'' or enter the back wormhole W_2, return to its past and propagate through the time machine once more. To fix the sector of coherency (classical alternative), we have to introduce projectors $1-Q_2$ and Q_2 corresponding to the two possibilities described above: escaping the time machine and entering it once more.

It is evident how this may be done. If the operator V_{20} describes the evolution between the time moments t'' (the final point of evolution) and t_2, then the operator

$$Q_2 = V_{20}^{-1} P_2 V_{20} = {V_{20}}^{\dagger} P_2 V_{20} \tag{19}$$

projects on those states at time t'' which after the evolution to the time t_2 will enter the region S_2 and return to the past. Accordingly, the operator $1-Q_2$ distinguishes the states corresponding to the particle escaping from the time machine.

Having the projector $1 - Q_2$ and using Eq. (17), one can find the partial evolution operators for the case $t' < t_1 < t'' < t_2$:

$$U_n = (1 - Q_2)K_n = (1 - Q_2)V_{01}(W_2 P_2 V_{21})^n (1 - P_1)V_1, \quad n \geq 0. \tag{20}$$

Let us remark that, in a sense, these operators, not (17), correspond to the topological classes of paths $\mathcal{P}_n$. Indeed, they actually fix the number of times the particle returns to its past, not only before time t'', but also after this. The operators (20) provide that the paths from the only one class $\mathcal{P}_n$ contribute the evolution of the particle after t'' as well as before this time.

Expressing Q_2 in (20) through P_2 (due to (19)) and using the equation $V_{21} = V_{20}V_{01}$ (multiplicativity for $V_{21} = V(t_2, t_1)$), we have finally the following formula for the partial evolution operators in the case $t' < t_1 < t'' < t_2$:

$$U_n = {V_{20}}^\dagger (1 - P_2)(V_{21} W_2 P_2)^n V_{21}(1 - P_1)V_1, \quad n \geq 0. \tag{21}$$

Let us try to prove the generalized unitarity for these partial evolution operators. One may for example substitute P_2 in the formula (20) by its expression through Q_2 to get

$$U_n = (1 - Q_2)V_{01}(W_2 V_{20} Q_2 V_{01})^n (1 - P_1)V_1, \quad n \geq 0. \tag{22}$$

Then, acting just as in Sect. 4, we have

$$\sum_{n=0}^{\infty} {U_n}^\dagger U_n = {V_1}^\dagger (1 - P_1)V_1. \tag{23}$$

The generalized unitarity is not yet fulfilled. The reason is that one more partial evolution operator must be considered, describing one more channel of evolution. Indeed, if the initial time is before t_1, then the particle have the possibility to enter the forward-directed wormhole W_1. The probability of this alternative is to be taken into account.

To take it into account, we must interpret the concept of a ‘final time moment’ in another way. So far we supposed that the ‘final time’ is nothing else than a space-like surface ($t'' = \text{const}$) *in the background spacetime* (the spacetime without wormholes).

However when the final time is within the time machine (i.e. later than t_1 but earlier than t_2) this 'time' may be thought of as a sum of the space-like surface in the background spacetime *plus* a space-like *slice of the wormhole* W_1.

With this wider concept of the final moment, the particle may arrive at this moment not only through the background spacetime but also through the wormhole W_1. The first possibility has been taken into account by the operators (22). The latter possibility is described by the operator

$$U_{00} = W_{01} P_1 V_1. \tag{24}$$

Adding the term ${U_{00}}^\dagger U_{00}$ to the sum (23) gives, in the case $t' < t_1 < t'' < t_2$, the generalized unitarity condition in the form of Eq. (10).

At last, consider the case $t_1 < t' < t'' < t_2$ when both initial t' and final t'' time moments are within the interval $[t_1, t_2]$. Using the partial evolution operators obtained from (18) by inclusion the projectors $1 - Q_2 = {V_{20}}^\dagger (1 - P_2) V_{20}$, we have

$$U_n = {V_{20'}}^\dagger (1 - P_2)(V_{21} W_2 P_2)^n V_{20}, \quad n \geq 0, \tag{25}$$

we shall easily prove the generalized unitarity in this case too:

$$\sum_{n=0}^{\infty} {U_n}^\dagger U_n = \mathbf{1}. \tag{26}$$

One more remark may be made in connection with the argument preceding Eq. (24). If the concept of time moment within the time machine is changed (as was discussed in the above-mentioned argument), the same wider concept must be applied not only to 'final', but also to the 'initial time moment'. Then, besides the operators (25), one more operator should be added,

$$U_{00} = W_{0'0} \tag{27}$$

describing evolution in the wormhole W_1 between two slices of this wormhole.

It seems at the first glance that this additional operator violates the (generalized) unitarity. One can see however that this is not the case. Indeed, by the unity operator $\mathbf{1}$

in Eq. (26), just as in all preceding formulas, we meant the unity operator in the space of states localized on the space-like surface of the background spacetime. If we express this in the notation explicitly, Eq. (26) should read

$$\sum_{n=0}^{\infty} {U_n}^{\dagger} U_n = \mathbf{1}_{\text{background}}. \tag{28}$$

The operator (27) satisfy the relation

$$ {U_{00}}^{\dagger} U_{00} = \mathbf{1}_{\text{wormhole}} \tag{29}$$

where the r.h.s. is the unity operator for the states localized on the space-like slice of the wormhole W_1. Summing up both preceding formulas, one has the generalized unitarity in the form

$$ {U_{00}}^{\dagger} U_{00} + \sum_{n=0}^{\infty} {U_n}^{\dagger} U_n = \mathbf{1}_{\text{complete}}. \tag{30}$$

Quite analogously, the operator of the form

$$U_{00} = V_2 W_{20} \tag{31}$$

may be added to the series of operators (16). The remarks analogous to those of the preceding paragraph may be made in this case. The generalized unitarity takes then the form (30) (instead of (15)) for the case $t_1 < t' < t_2 < t''$ too.

Thus, the generalized unitarity takes place for an arbitrary time interval. Therefore, the concept of probability may be used correctly even within the time machine.

Let us make one more remark. We saw that an additional projecting factor $1 - Q_2$ should be including in the partial evolution operators U_n corresponding to the final time within the time machine, $t_1 < t'' < t_2$. The same logic seems to require inclusion of an analogous factor $1 - Q_1$ in the expressions for U_n in the case when the initial time is within the time machine, $t_1 < t' < t_2$. This actually may be done. Then we guarantee that the operators U_n describe propagation of only those states that have resulted from the evolution starting earlier than t_1. This however is not necessary because the operators

U_n may be quite reasonably applied to arbitrary states prepared at time t' within the time machine. This is why the projector $1 - Q_1$ turned out to be not necessary for providing the generalized unitarity.

Preparation of the state within the time machine could be restricted by the condition of self-consistency if interaction of the particle with its duplicates were taken into account. We however consider non-interacting particles (i.e. we suppose that the interaction is negligible). Therefore, the state of *one particle* at time t' within the time machine may be prepared arbitrarily.[4] Further evolution of this arbitrary state will be described by one of the partial evolution operators U_n. Any number n may be realized in this evolution, the probability of each of them being determined by the formula (11).

6 Multiplicativity for the Time Machine

Let us address now the question of multiplicativity for the evolution in the time-machine spacetime. Multiplicativity of the propagators or evolution operators (3) means that the evolution during some time interval may be considered in two stages as evolution during two subintervals. With the decoherence caused by measurement, the multiplicativity is described by the formula (8). We should now try and interpret this formula for the evolution in the time-machine spacetime with the specific decoherence (superselection) arising in this case.

The winding number n introduced in Sect. 3 through the classes of paths $\mathcal{P}_n$ hints how multiplicativity might be defined and proved in the case of the time machine. Indeed, this number is equal to the number of times the particle returns to its past travelling through the wormhole W_2. If we have two alternatives characterized by the numbers n_1

[4]Notice that the complete description of the states of all particles at this moment (including the original particle and its duplicates) is not arbitrary. Indeed, if we are preparing, at time t' between moments t_1 and t_2, the state that will n times return to its past, then it is already known at t' that n duplicates of the particle exist coming from the region S_1 at time t_1.

and n_2 correspondingly, then the product of these alternatives corresponds to the number $n = n_1 + n_2$. This means simply that if the particle returns n_1 times and then once more returns n_2 times to its past, then ultimately it returns n_1+n_2 times. It is evident that the (generalized) multiplicativity (8) might have, in the case of a time machine, the following form that can be readily proved for the operators K_n:

$$K_{00}(t'',t')\,K_{00}(t',t) = K_{00}(t'',t), \quad K_m(t'',t')\,K_n(t',t) = K_{m+n}(t'',t), \quad n \geq 0. \tag{32}$$

The relation (32) may be readily proved for the operators K_n with the help of the path-integral representation (9). The proof is based on the fact that each path $p \in \mathcal{P}_{m+n}$ may be presented as a product of the paths $p_1 \in \mathcal{P}_m$ and $p_2 \in \mathcal{P}_n$, and vice versa, the product of any paths $p_1 \in \mathcal{P}_m$, $p_2 \in \mathcal{P}_n$ gives a path belonging to $\mathcal{P}_{m+n}$. By 'product' we mean passing through one of the paths and then through another one.The product of two paths is defined only if the first path ends in the point where the second one starts (see [18], Chapter 10 for details of this algebra of paths).

Instead of this, the relation (32) for operators K_n may easily be proved by using explicit expressions for these operators derived in Sect. 4. The latter way of consideration may be applied also to the partial evolution operators U_n which, as we know, do not always coincide with the operators K_n. We shall see that for partial evolution operators U_n the second of the relations (32) is valid only when at least one of the numbers m, n is zero.

It is evident that the generalized multiplicativity (32) is valid in the trivial cases $t' < t'' < t_1$ and $t_2 < t' < t''$, when it reduces to the conventional multiplicativity because both $m = n = 0$:

$$U_0(t'',t')\,U_0(t',t) = U_0(t'',t).$$

Trivial is also the case when one of the intervals $[t,t']$ or $[t',t'']$ lies in one of the chronal regions. Then the multiplicativity takes one the following forms that can be easily proven:

$$U_n(t'',t) = U_0(t'',t')\,U_n(t',t) \quad \text{or} \quad U_n(t'',t) = U_n(t'',t')\,U_0(t',t).$$

Consider now non-trivial situations.

Summing up the results of Sects. 4, 5, we have the following formulas for the partial evolution operators $U_{00}(t'', t')$, $U_n(t'', t')$ (where $n \geq 0$):

$$
\begin{aligned}
&t' < t_1 < t_2 < t'' : \\
&\qquad U_{00} = V_2 W_1 P_1 V_1, \quad U_n = V_2(1-P_2)(V_{21}W_2P_2)^n V_{21}(1-P_1)V_1. \\
&t_1 < t' < t_2 < t'' : \\
&\qquad U_{00} = V_2 W_{20}, \quad U_n = V_2(1-P_2)(V_{21}W_2P_2)^n V_{20}. \\
&t' < t_1 < t'' < t_2 : \\
&\qquad U_{00} = W_{01} P_1 V_1, \quad U_n = {V_{20}}^{\dagger}(1-P_2)(V_{21}W_2P_2)^n V_{21}(1-P_1)V_1. \\
&t_1 < t' < t'' < t_2 : \\
&\qquad U_{00} = W_{0'0}, \quad U_n = {V_{20'}}^{\dagger}(1-P_2)(V_{21}W_2P_2)^n V_{20}.
\end{aligned}
$$

One can verify straightforwardly that, of all multiplicativity relations (32), the following are valid also for the partial evolution operators:

$$U_{00}(t'', t')\, U_{00}(t', t) = U_{00}(t'', t), \quad U_0(t'', t')\, U_n(t', t) = U_n(t'', t), \quad n \geq 0. \tag{33}$$

If $m \neq 0$, the second of the relations (32) is not valid for U_n. Moreover, if we defined the partial evolution operators with the time argument within the time machine in such a way as it was supposed in the remark at the end of Sect. 5, then even the products of the form $U_0\, U_n$ with $n \neq 0$ would also give zero.

We see therefore that a topologically non-trivial evolution (described by the evolution operator U_n with $n \geq 1$) cannot be presented as the product of two topologically non-trivial evolutions. Topologically non-trivial evolution is 'integral in time', it cannot be followed step by step: first $m \neq 0$ returns to the past, then again $n \neq 0$ returns that finally gives $m + n$ returns.

This is not astonishing. If we describe the evolution within the time machine, not achieving an exit from it, then this part of evolution cannot be characterized by a definite

number n of returns of the particle to the past (this number depends on the subsequent stage of evolution). In a sense, evolution within the time machine is not local in time, influence of the future cannot be excluded.

We succeeded in constructing operators describing evolution to the time moment t'' within the time machine, for example U_n(in,past). However a special projector in this operator provides escaping from the time machine after time t''. This operator guarantees that the future stage of evolution (after t'') will not influence the past evolution (until t''). Thus, non-locality of evolution within the time machine is evident even from the structure of partial evolution operators, not only from the form of multiplicativity for them.

On the other side, the operators K_n (but not U_n) possess the property of multiplicativity (32) for arbitrary time arguments.

One may object that expressing multiplicativity in terms of the operators K_n, in fact non-physical within the time machine, makes no sense (if the superselection in n takes place). In fact, each of these operators describes propagation of non-physical states (forbidden superpositions). However, the product of such operators,

$$K_{n_N}(t'', \tau_{N-1}) \ldots K_{n_2}(\tau_2, \tau_1) K_{n_1}(\tau_1, t') = K_{n_1+n_2+\ldots+n_N}(t'', t'),$$

with $t' < t_1 < t_2 < t''$, is a correct partial evolution operator. No non-physical state is propagated due to this operator. Thus, even though each of the operators K_n describes propagation of non-physical states within the time machine, the result of the propagation from the past (in respect to the time machine) to the future of the time machine is presented by these operators correctly.

One may say that the operators K_n give multiplicative description of evolution even within the time machine, but at the price of introducing non-physical states in intermediate stages.

One more interpretation of the results obtained is complementarity between (generalized) unitarity and multiplicativity. One may describe the evolution within the time machine by the operators U_n (unitary in the generalized sense but satisfying only trivial

multiplicativity relations) or by K_n (satisfying the generalized multiplicativity but not unitarity).

7 Comparison with the Unitary Theory

We showed in Sects. 4, 5 that the description of the particle evolution in the time-machine spacetime with the help of the partial propagators (partial evolution operators) is correct from the point of view of probabilities. This statement means that the partial evolution operators U_n satisfy the condition of generalized unitarity (10).

In the conventional theory, when no decoherence (superselection) is supposed, the evolution is described by a single evolution operator U which must be unitary, $U^\dagger U = 1$. In a number of papers evolution of particles in a time-machine spacetime was described in this way (see [11] and references therein). Let us compare two types of theories. This is not difficult because we used here the same model of spacetime as in [11].

It was shown in [11] that a single evolution operator $U = U(t'', t')$ is unitary in the case $t' < t_1 < t_2 < t''$ when the evolution from the past of the time machine to its future is considered. At the first glance, this contradicts to our conclusion about generalized unitarity of the partial evolution operators U_n in this case.

Indeed, the evolution operator U was defined in [11] by summation over all paths. In our terms, this means that this operator is equal to

$$U = U_{00} + \sum_{n=0}^{\infty} U_n \tag{34}$$

where U_n are defined by Eq. (12). Unitarity for this operator means that

$${U_{00}}^\dagger U_{00} + \sum_{n=0}^{\infty} {U_{00}}^\dagger U_n + \sum_{n=0}^{\infty} {U_n}^\dagger U_{00} + \sum_{m,n=0}^{\infty} {U_m}^\dagger U_n = \mathbf{1}. \tag{35}$$

The formulas (10) and (35) differ by the non-diagonal terms (they are absent in the former case). Both formulas may be valid only if a sum of non-diagonal terms is zero. It turns out however that this actually takes place in the considered case ($t' < t_1 < t_2 < t''$), so

that both forms of unitarity turn out to be valid. Let us show this with the evolution operators (12).

Consider an off-diagonal term in (35), ${U_{n+k}}^\dagger U_n$, $k \geq 1$. Using Eq. (12), one may present this term in the form

$$ {U_{n+k}}^\dagger U_n = A^\dagger (B^\dagger)^{n+k-1} C B^{n-1} A - A^\dagger (B^\dagger)^{n+k} C B^n A \tag{36} $$

where it is denoted

$$ A = (1-P_1)V_1, \quad B = W_2 P_2 V_{21}, \quad C = {V_{21}}^\dagger P_2 V_{21}. $$

The formula (36) is valid for all $n \geq 1$. Summing up this expression in n from 1 to ∞ and adding the terms ${U_{k-1}}^\dagger U_{00}$ and ${U_k}^\dagger U_0$, we have

$$ \begin{aligned} & {U_{k-1}}^\dagger U_{00} + \sum_{n=0}^{\infty} {U_{n+k}}^\dagger U_n \\ = \;& {V_1}^\dagger (1-P_1)({V_{21}}^\dagger P_2 {W_2}^\dagger)^{k-1} {V_{21}}^\dagger (1-P_2) W_1 P_1 V_1 \\ + \;& {V_1}^\dagger (1-P_1)({V_{21}}^\dagger P_2 {W_2}^\dagger)^{k} {V_{21}}^\dagger (1-P_2) V_{21} (1-P_1) V_1 \\ + \;& {V_1}^\dagger (1-P_1)({V_{21}}^\dagger P_2 {W_2}^\dagger)^{k} {V_{21}}^\dagger P_2 V_{21} (1-P_1) V_1 \end{aligned} $$

Two last terms here can be summed up to give a more simple expression:

$$ \begin{aligned} & {U_{k-1}}^\dagger U_{00} + \sum_{n=0}^{\infty} {U_{n+k}}^\dagger U_n \\ = \;& {V_1}^\dagger (1-P_1)({V_{21}}^\dagger P_2 {W_2}^\dagger)^{k-1} {V_{21}}^\dagger (1-P_2) W_1 P_1 V_1 \\ + \;& {V_1}^\dagger (1-P_1)({V_{21}}^\dagger P_2 {W_2}^\dagger)^{k} (1-P_1) V_1 \end{aligned} \tag{37} $$

In the r.h.s. of the latter expression however both terms are equal to zero. The reason is the following equations:

$$ (1-P_2)W_1 = 0, \quad (1-P_1)W_2 = 0. \tag{38} $$

The first of them is a consequence of the fact that after propagation in the wormhole W_1 the particle is in the space region S_2. The operator P_2 projects on this region, and

$1 - P_2$ projects on the complementary region (all the space-like surface $t_2 = \text{const}$ of the background spacetime excluding the region S_2). Going out of the wormhole W_1 the particle cannot be in this complementary region. Therefore, the first equation in (38) is valid. Analogously, the second equation in (38) follows from the fact that after travelling in the wormhole W_2 the particle turns out to be in the space region S_1, and projecting, by $1 - P_1$, on the complementary region gives zero.

Using the first equation from (38) and a conjugate to the second equation we see that the expression (37) as well as its conjugate are equal to zero. Therefore,

$$U_{k-1}{}^{\dagger} U_{00} + \sum_{n=0}^{\infty} U_{n+k}{}^{\dagger} U_n = U_{00}{}^{\dagger} U_{k-1} + \sum_{n=0}^{\infty} U_n{}^{\dagger} U_{n+k} = 0. \tag{39}$$

Thus, the off-diagonal terms do not contribute the sum (35), and both unitarity and generalized unitarity are valid in the considered case: evolution from the initial chronal region (earlier than the time machine) to the final chronal region (after it).

This does not mean of course that both descriptions, one with the superselection and one without superselection, are equivalent. According to the coherent description, the evolution law is following:

$$\rho' = U\,\rho\,U^{\dagger} = U_{00}\,\rho\,U_{00}{}^{\dagger} + \sum_{n=0}^{\infty} U_{00}\,\rho\,U_n{}^{\dagger} + \sum_{n=0}^{\infty} U_n\,\rho\,U_{00}{}^{\dagger} + \sum_{m,n=0}^{\infty} U_m\,\rho\,U_n{}^{\dagger}.$$

According to the theory with decoherence (superselection), the evolution should be described by partial evolution operators according to a general formula (5) that takes the following form in our case:

$$\rho' = U_{00}\,\rho\,U_{00}{}^{\dagger} + \sum_{n=0}^{\infty} U_n\,\rho\,U_n{}^{\dagger}. \tag{40}$$

This leads of course to different predictions. Thus, the two theories have different physical contents.[5]

[5] Let us repeat once more that decoherence of different n leading to generalized unitarity must take place "in ordinary conditions" while coherent description may be valid "in special conditions" when the environment does not distinguish between different numbers of particles in the dischronal region.

Consider different locations of times t', t'' (the start and the end of the evolution) in respect to the time machine. In all cases when at least one of the time moments t', t'' is within the time machine, no unitarity is found in the paper [11]. However, using the definition (20), we can prove Eq. (39) in just the same way as above for the case $t' < t_1 < t'' < t_2$. Thus, in this case operator (34) is also unitary. There is again no contradiction with the result of [11] because in that paper not this operator, but $\sum_n K_n$ was considered as an evolution operator (both definitions coincide for $t' < t_1 < t_2 < t''$).

Formally unitarity for $t' < t_1 < t'' < t_2$ is caused by the projector $1 - Q_2$ in the definition of U_n. Physically the reason of this is that this projector provides escaping the particle from the time machine (see Sect. 5). Thus, though we deal with $t'' < t_2$, actually the situation is, in a sense, equivalent to the case $t'' > t_2$.

For the cases $t_1 < t' < t_2 < t''$ and $t_1 < t' < t'' < t_2$ the relation $U^\dagger U = 1$ is invalid, and we are left only with the generalized unitarity. We could introduce in these cases too a sort of projector providing unitarity of the operator $\sum_n U_n$. However there is no physical ground for this (see the remark at the end of Sect. 5).

8 States Trapped in the Time Machine

We considered the evolution of a particle entering the time machine from the past chronal region, circulating several times within the time machine and then escaping from it into the future chronal region. There are however the states that exist only within the time machine but not in the chronal regions. These states may be described at arbitrary time moment between t_1 and t_2. We shall describe them at time t_1.

Consider the state $|\psi_1\rangle$ at time t_1 possessing the properties

$$(1 - P_2)V_{21}|\psi_1\rangle = 0, \quad (1 - W_2 V_{21})|\psi_1\rangle = 0$$

It is evident that the particle in this state "bites its own tail": after evolution to t_2 it enters the wormhole W_2 and after going out of the wormhole it turns out to be in the

state identical with the initial state. After this the same cycle of evolution is repeated. The particle in such a state travels through the interval $[t_1, t_2]$, but it never passes any time moment before t_1 or any time moment after t_2.

There is evident generalization of this state, providing passing the dischronal region twice, three times or generally n times before reaching the initial state. Such a state satisfies the following conditions at time t_1:

$$\begin{aligned} (1 - P_2)\,(V_{21}W_2)^{n'} V_{21}|\psi_n\rangle &= 0 \text{ for } n' < n, \\ (1 - (W_2V_{21})^n)|\psi_n\rangle &= 0. \end{aligned} \tag{41}$$

Then the particle "bites its own tail" after winding n causal loops: after n cycles, each of which contains travelling through the interval $[t_1, t_2]$, entering the wormhole W_2 and passing this wormhole, the state is identical to the initial one.

The states thus described are nothing else than a quantum version of the classical "Jinnee" discussed in [19, 20], the classical bodies moving along closed time-like trajectories within a time machine.

One can construct a state in such a way as to provide passing the dischronal region infinite number of times never escaping to the future, but with the initial state never being repeated. The condition for this is (at time t_1)

$$(1 - P_2)\,(V_{21}W_2)^{n'} V_{21}|\psi_\infty\rangle = 0 \text{ (all } n'). \tag{42}$$

The particle in such a state cannot go out of the time machine though probably its state never becomes identical to the initial state. The states (41) may be considered to be special cases of (42).

One more generalization can be considered: the state existing earlier than the time machine emerged (before t_1) but trapped in TM so that it cannot be found at times later than t_2. Let us characterize this state at time $t' < t_1$:

$$\begin{aligned} P_1V_1|\psi_n\rangle &= 0 \text{ (the particle does not enter } W_1), \\ (1 - P_2)(V_{21}W_2)^{n'} V_{21}(1 - P_1)V_1|\psi_n\rangle &= 0, \quad n' \geq 1 \text{ (it does not escape TM).} \end{aligned}$$

It may be shown that the state of the particle in this case cannot be repeated after a finite number n of cycles. The condition for such a repetition,

$$(1-(V_{21}W_2)^n)V_{21}(1-P_1)V_1|\psi_n\rangle = 0 \text{ (the state is repeated after } n \text{ cycles).}$$

would be inconsistent due to the relation $W_2 = P_1 W_2$ (expressing that the mouth leading out from the wormhole W_1 is in the region S_1).

Therefore, it is possible that a particle enters the time machine from the past and stays within it, infinitely repeating evolution from t_1 to t_2 and backward. Moreover, the states obtained by time reversal from those already considered, may also be defined. Then the particle escapes from the time machine into the future after infinite number of cycles within the time machine, never being in the past chronal region.

Existence of the states trapped in the time machine contradicts to the generalized unitarity condition in the form given earlier. Indeed, this condition provides that any state evolves finally to the future chronal region through one of the channels enumerated by n. The trapped states do not at all escape from the time machine. These states belong to the class of 'finely tuned' states which the generalized unitarity condition is not valid for (see Remark 1 at page 17).

For example, application of the (generalized) unitarity condition (26) for evolution from the time t_1 (considered as a time within the time machine) to the same time t_1 should seemingly forbid the trapped states. However, analysing attentively the proof of this condition, we see that it is supposed in this proof that the operator

$$I_N = 1 - \sum_{n=0}^{N} U_n{}^{\dagger}\, U_n = \left(V_{21}{}^{\dagger}\, P_2\, W_2{}^{\dagger}\right)^N V_{21}{}^{\dagger}\, P_2\, V_{21}\, (W_2\, P_2\, V_{21})^N$$

tends to zero when $N \to \infty$. The expectation value of this operator for almost all states (as well as the trace of this operator multiplied by almost all density matrices) decreases with increasing N. However, for the states considered in the present section (the trapped states) we have, as a consequence of Eq. (42),

$$\langle\psi|\, 1 - \sum_{n=0}^{N} U_n{}^{\dagger}\, U_n\, |\psi\rangle = \langle\psi|\psi\rangle = 1.$$

Hence the operator I_N does not decrease and the generalized unitarity condition is not fulfilled in the space of the trapped states.

The physical interpretation of this fact in terms of probabilities is following. For this very special class of states (not escaping to the future chronal region) no part of probability goes into channels enumerated by n. It might be reasonable to introduce one more channel corresponding to $n = \infty$ and claim that all probability, for these states, goes into this channel. This leads to the following remark.

Remark 2 In a general case (for arbitrary states) the generalized unitarity condition should be written in the form

$$U_{00}{}^\dagger U_{00} + \sum_{n=0}^{N} U_n{}^\dagger U_n + I_N = \mathbf{1}$$

(where the operator I_N is given above for the case $t' = t'' = t_1$ and can be readily written for all other choices of t' and t''). The expectation value of this formula for an arbitrary state $|\psi\rangle$,

$$\langle\psi|U_{00}{}^\dagger U_{00}|\psi\rangle + \sum_{n=0}^{N}\langle\psi|U_n{}^\dagger U_n|\psi\rangle + \langle\psi|I_N|\psi\rangle = 1,$$

gives the probability distribution for different channels of evolution (the last term in the l.h.s. gives the probability for all channels with numbers more than N). If the limit

$$\langle\psi|I_\infty|\psi\rangle = \lim_{N\to\infty}\langle\psi|I_N|\psi\rangle$$

exists, the formula

$$\langle\psi|U_{00}{}^\dagger U_{00}|\psi\rangle + \sum_{n=0}^{\infty}\langle\psi|U_n{}^\dagger U_n|\psi\rangle + \langle\psi|I_\infty|\psi\rangle = 1$$

gives the probability distribution for all channels with finite n and for the channel with $n = \infty$ corresponding to the trapped states.

9 Conclusion

We considered in this paper how evolution of a non-relativistic non-interacting quantum particle in the spacetime with closed time-like curves (a time-machine spacetime) should be described. A simple non-relativistic model of such a spacetime was used corresponding to emergence of the time machine at some time moment and its disappearance at another moment.

In the paper [11] evolution of a particle in this spacetime was described with the help of the evolution operator U found by path integration. It was shown that such an operator is multiplicative and unitary only when the evolution is considered between the time moments belonging to the chronal regions (including the case when one of the time moments is in the past chronal region and the other is in the future chronal region).

In our paper the arguments were given that superselection (decoherence) may arise for the evolution in this spacetime (in "normal conditions"). The superselection sectors are enumerated in this case by the number of times the particle returns to its past. Therefore, evolution of the particle is described by a family of *partial evolution operators* U_n (or partial propagators) instead of a single evolution operator U.

In the case when the evolution is considered into the future chronal region (i.e. to the time moment after the time machine disappearance), the partial propagators may be correctly defined as integrals over the corresponding topological classes of paths. If the final time of evolution is within the time machine, the path-integral expressions for the partial propagators (evolution operators) should be corrected by the projectors providing escaping the particle from the time machine after a certain number of returns to the past.

With these definitions, the family of partial evolution operators satisfy the generalized unitarity condition $\sum_n {U_n}^\dagger U_n = 1$ and multiplicativity condition $U_n U_m = U_{n+m}$ (the latter is fulfilled only for at least one of the numbers n, m equal to zero). It is shown that the generalized unitarity is compatible with the unitarity of the operator $U = \sum_n U_n$ in the case of evolution from the past of the time machine to its future. Nevertheless, the

operator U cannot describe correctly the evolution if the superselection exists.

A special class of states of the particle is described emerging and disappearing simultaneously with the time machine, i.e. existing only within it. These states present a quantum analogue of the classical "Jinnee" states investigated earlier [19, 20]. Besides this, the states are considered that either 1) exist in the past of the time machine but then are trapped in it and never escape into the future of the time machine, or 2) never exist in the past of the time machine, have infinite number of cycles within the time machine and then finally escape into its future. For all these states the generalized unitarity condition should be corrected to take into account the channel of evolution with $n = \infty$.

From conceptual point of view, the superselection of the considered type may be thought of as the influence of the future on the past, i.e. a sort of "consistency conditions".

The coherent (unitary) evolution in the time-machine spacetime is also possible, but only in quite special conditions when the environment of the particle in the dischronal region does not distinguish between different numbers of particles.[6] It is interesting that in the case of coherent evolution no information from the future to the past can be carried by the particle. If the particle leaves some information about the future in its past, then the interaction responsible for the information transfer leads to decoherence so that the winding number n becomes definite.

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[6]This question needs special investigation with a concrete model of an environment. A very good discussion of the role of environment in different situations may be found in [21].

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